Introduction to Differentials

David G Radcliffe

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1 Increments

Let y be a function of x, say $y = f(x)$. The symbol Δx denotes a change or increment in the value of x. Note that a change in the value of x will produce a corresponding change in the value of y. More precisely, if the value of x changes from x_1 to x_2, then the value of y changes from $f(x_1)$ to $f(x_2)$.

$$\Delta x = x_2 - x_1$$
$$\Delta y = f(x_2) - f(x_1)$$

Example: Let x and y be related by the equation $y = x^2$. If x decreases from 10 to 8, then calculate Δx and Δy.

Solution:

$$\Delta x = 8 - 10 = -2$$
$$\Delta y = 8^2 - 10^2 = -36$$

Suppose that $f(x)$ is differentiable at x_1. Then

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\,.$$

This implies that

$$\Delta y \approx f'(x_1)\,\Delta x$$

provided that Δx is sufficiently small. (The symbol $\approx$ means *approximately equals.*)

2 Differentials

Differentials are similar to increments, but more subtle. The differential of x is denoted dx.

We need to distinguish between two types of variables: *dependent* and *independent*. A variable is *dependent* if is a function of one or more other variables, but it is *independent* if is not a function of another variable.

If x is an independent variable, then dx and Δx are identical. On the other hand, if $y = f(x)$, then

$$dy = f'(x)\,dx.$$

In this equation, dx is an independent variable, but dy is a dependent variable, since dy is a function of both x and dx.

2.1 Properties of differentials

In the following, u and v are functions, and c and n are constants.

- $d(c) = 0$
- $d(u + v) = du + dv$
- $d(u - v) = du - dv$
- $d(cu) = c\,du$
- $d(uv) = v\,du + u\,dv$
- $d(u/v) = (v\,du - u\,dv)/v^2$
- $d(u^n) = nu^{n-1}\,du$
- If $y = f(u)$ then $dy = f'(u)\,du$

2.2 Exercises

Calculate each differential. (Cover up the answers in the right column.)

$d(x + y - z) =$	$dx + dy - dz$
$d(17x + 8y + 25) =$	$17\,dx + 8\,dy$
$d(rs) =$	$s\,dr + r\,ds$
$d(xyz) =$	$yz\,dx + xz\,dy + xy\,dz$
$d(x^{43}) =$	$43x^{42}\,dx$
$d(\pi r^2) =$	$2\pi r\,dr$
$d(\frac{1}{3}\pi r^2 h) =$	$\frac{1}{3}\pi(2rh\,dr + r^2\,dh)$
$d(x \ln y) =$	$(\ln y)\,dx + \frac{x}{y}\,dy$
$d(\cos(x^2 y)) =$	$-\sin(x^2 y) \cdot (2xy\,dx + x^2\,dy)$

3 Differentials as estimates

Suppose that $y = f(x)$, where f is a differentiable function. Then dy is approximately equal to Δy, provided that dx is sufficiently small. Therefore, we can use the differential dy to *estimate* the change in y.

Example: Suppose that x and y are related by the equation $y = 2x^3$. If x increases from 10 to 10.2, then what is the estimated change in y?

Solution:

$$dy = 6x^2\,dx = 6(10)^2(0.2) = 120$$

Example: Estimate the change in the area of a rectangle, if the length increases from 40 to 41, and the width increases from 15 to 17.

Solution:

$$\begin{aligned} A &= LW \\ dA &= W\,dL + L\,dW \\ dA &= 15 \cdot 1 + 40 \cdot 2 = 95 \end{aligned}$$

3.1 Exercises

1. Estimate the change in the area of a triangle, if the base increases from 12 to 12.2, and the height decreases from 10 to 9.7. (Use $A = \frac{1}{2}bh$.)
2. Estimate the change in the volume of a cylinder, if the radius increases from 30 to 30.2, and the height decreases from 10 to 9.99. (Use $V = \pi r^2 h$.)
3. Estimate the change in the surface area of a sphere, if the radius decreases from 10 to 9.5. (Use $A = 4\pi r^2$.)
4. Suppose that x, y, z are related by the equation $z = x^2y^3$. If x increases from 5 to 5.01, and y increases from 10 to 10.03, then estimate the change in z.

Answers: -0.8, 111π, -40π, 325

4 Using differentials to estimate roots

We can use differentials to estimate square roots and n^{th} roots without using a calculator. Suppose that we wish to estimate the square root of 26.2. We observe that 26.2 is close to 25, and we already know that the square root of 25 is 5. So, we begin with the equation $y = x^{1/2}$, and we estimate the change in y when x is increased from 25 to 26.2.

$$
\begin{aligned}
dy &= \frac{1}{2}x^{-1/2}\,dx \\
dy &= \frac{1}{2}25^{-1/2}\,(1.2) = 0.12 \\
y + dy &= 5.12
\end{aligned}
$$

If x increases from 25 to 26.2, then y increases by approximately 0.12, so the estimated value of $\sqrt{26.2}$ is 5.12.

We can use the same method to approximate the values of other kinds of functions.

4.1 Exercises

1. Estimate $\sqrt{50}$ given that $\sqrt{49} = 7$.
2. Estimate $\sqrt[3]{25}$ given that $\sqrt[3]{27} = 3$.
3. Estimate $\sqrt{164}$.
4. Estimate $\sin\left(\frac{\pi}{3} + 0.02\right)$.

Answers: $7 + 1/14$, $\quad 3 - 2/27$, $\quad 13 - 5/26$, $\quad \dfrac{\sqrt{3}}{2} + 0.01$.

5 Relative change

If the value of x is increased by Δx, then the *relative change* in x is $\Delta x/x$. If y is a differentiable function of x, then the relative change in y is approximately dy/y, provided that dx is sufficiently small. The expression dy/y is called a *relative differential*, or a *logarithmic differential*. Relative differentials are often expressed as percentages.

5.1 Properties of relative differentials

- If $y = cx$ then $\dfrac{dy}{y} = \dfrac{dx}{x}$
- If $y = uv$ then $\dfrac{dy}{y} = \dfrac{du}{u} + \dfrac{dv}{v}$
- If $y = u/v$ then $\dfrac{dy}{y} = \dfrac{du}{u} - \dfrac{dv}{v}$
- If $y = u^n$ then $\dfrac{dy}{y} = n \cdot \dfrac{du}{u}$

Example: The side of a square increases by 0.2%. Estimate the percentage increase in the area of the square.

Solution:

$$\begin{aligned} A &= s^2 \\ \frac{dA}{A} &= 2\frac{ds}{s} \\ \frac{dA}{A} &= 2(0.2\%) = 0.4\% \end{aligned}$$

Example: The variables x, y, z are related by the equation $z = x\sqrt{y}$. If x increases by 2% and y increases by 3%, estimate the percentage change in z.

Solution:

$$\begin{aligned} z &= xy^{1/2} \\ \frac{dz}{z} &= \frac{dx}{x} + \frac{d(y^{1/2})}{y^{1/2}} \\ \frac{dz}{z} &= \frac{dx}{x} + \frac{1}{2}\cdot\frac{dy}{y} \\ \frac{dz}{z} &= 2\% + \frac{1}{2}(3\%) = 3.5\% \end{aligned}$$

5.2 Exercises

1. The length of a rectangle increases by 2% and the width increases by 3%. What is the estimated percentage change in the area of the rectangle?

2. The side of a cube increases by 0.1%. What is the estimated percentage change in the volume and the surface area of the cube?

3. What is the estimated change in the volume of a cone, if the radius increases by 0.5% and the height increases by 1%? (Use $V = \frac{1}{3}\pi r^2 h$.)

4. The variables x, y, and z are related by the equation $z = \dfrac{13x^2}{y^3}$. If x increases by 0.15% and y decreases by 0.2% then what is the estimated percentage change in z?

Answers: 5%, 0.3% and 0.2%, 2%, 0.9%